

An ω -categorical structure with amenable automorphism group

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Abstract. We analyse ω -categorical precompact expansions of particular ω -categorical structures from the viewpoint of amenability of their automorphism groups.

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0 Introduction

A group G is called **amenable** if every G -flow (i.e. a compact Hausdorff space along with a continuous G -action) supports an invariant Borel probability measure. If every G -flow has a fixed point then we say that G is **extremely amenable**. Let M be a relational countably categorical structure which is a Fraïssé limit of a Fraïssé class \mathcal{K} . In particular \mathcal{K} coincides with $\text{Age}(M)$, the class of all finite substructures of M . By Theorem 4.8 of [9] the group $\text{Aut}(M)$ is *extremely amenable if and only if the class \mathcal{K} has the Ramsey property and consists of rigid elements*. Here the class \mathcal{K} is said to have the Ramsey property if for any k and a pair $A < B$ from \mathcal{K} there exists $C \in \mathcal{K}$ so that each k -coloring

$$\xi : \binom{C}{A} \rightarrow k$$

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is monochromatic on some $\binom{B'}{A'}$ from C which is a copy of $\binom{B}{A}$, i.e.

$$C \rightarrow (B)_k^A.$$

We remind the reader that $\binom{C}{A}$ denotes the set of all substructures of C isomorphic to A . This result has become a basic tool to amenability of automorphism groups. To see whether $\text{Aut}(M)$ is amenable one usually looks for an expansion M^* of M so that M^* is a Fraïssé structure with extremely amenable $\text{Aut}(M^*)$. Moreover it is usually assumed that M^* is a **precompact** expansion of M , i.e. every member of \mathcal{K} has finitely many expansions in $\text{Age}(M^*)$, see [9], [10], [12], [1] and [13]. Theorem 9.2 from [1] and Theorem 2.1 from [13] describe amenability of $\text{Aut}(M)$ in this situation. The question if there is a countably categorical structure M with amenable automorphism group which does not have expansions as above was formulated by several people. We mention very similar Problems 27, 28 in [2] where precompactness is replaced by ω -categoricity and finite homogeneity.

We think that in order to construct a required example one can use the ideas applied in [7] where we construct an ω -categorical structure so that its theory is not G-compact and it does not have AZ-enumerations. These ideas develop ones applied in slightly different forms in [8] and [6] for some other questions. Moreover Casanovas, Pelaez and Ziegler suggest in [3] a general method which simplifies and generalises our approach from [6], [7] and [8]. The basic object of this construction is a particular theory T_E of equivalence relations E_n on n -tuples. The paper [3] pays attention to several model-theoretic properties of T_E .

Below we study T_E from the viewpoint of (extreme) amenability of its expansions. Then we apply our results to a construction of a family of concrete candidates for an example of an ω -categorical structure with amenable automorphism group and without ω -categorical precompact expansions with extremely amenable automorphism groups. We will in particular show that these structures have the following unusual combination of properties:

- the automorphism group is amenable;
- it does not satisfy Hrushovski's extension property;
- it does not have an order expansion with the Ramsey property.

In fact we will show a slightly stronger version of the latter property.

1 Equivalence relations

We start with a very interesting reduct of the structure from [7]. This is T_E mentioned in the introduction. It has already deserved some attention in model-theoretic community, see [3].

Let $L_0 = \{E_n : 0 < n < \omega\}$ be a first-order language, where each E_n is a relational symbol of arity $2n$. Let \mathcal{K}_0 be the class of all finite L_0 -structures C where each relation

$E_n(\bar{x}, \bar{y})$ determines an equivalence relation on the set (denoted by $\binom{C}{n}$) of unordered n -element subsets of C . In particular for every n the class \mathcal{K}_0 satisfies the sentence

$$\forall \bar{x} \bar{y} (E_n(x_1, \dots, x_n, y_1, \dots, y_n) \rightarrow \bigwedge \{E_n(y_1, \dots, y_n, x_{\sigma(1)}, \dots, x_{\sigma(n)}) : \sigma \in \text{Sym}(n)\}).$$

Note that for $C \in \mathcal{K}_0$, E_n is not satisfied by \bar{a}, \bar{b} if one of these tuples has a repetition. Thus for $n > |C|$ we put that no $2n$ -tuple from C satisfies $E_n(\bar{x}, \bar{y})$. It is easy to see that \mathcal{K}_0 is closed under taking substructures and the number of isomorphism types of \mathcal{K}_0 -structures of any size is finite.

Let us verify *the amalgamation property for \mathcal{K}_0* . Given $A, B_1, B_2 \in \mathcal{K}_0$ with $B_1 \cap B_2 = A$, define $C \in \mathcal{K}_0$ as $B_1 \cup B_2$ with the finest equivalence relations among those which obey the following rules. When $n \leq |B_1 \cup B_2|$ and $\bar{a} \in \binom{B_1}{n} \cup \binom{B_2}{n}$ we put that the E_n -class of \bar{a} in C is contained in $\binom{B_1}{n} \cup \binom{B_2}{n}$. We also assume that all n -tuples meeting both $B_1 \setminus B_2$ and $B_2 \setminus B_1$ are pairwise equivalent with respect to E_n . In particular if $n \geq \max(|B_1|, |B_2|)$ we put that all n -element n -tuples from C are pairwise E_n -equivalent.

It is easy to see that this amalgamation also works for the joint embedding property.

Let M_0 be the countable universal homogeneous structure for \mathcal{K}_0 . It is clear that in M_0 each E_n defines infinitely many classes and each E_n -class is infinite. Let $T_E = \text{Th}(M_0)$.

Theorem 1.2 which we prove below, shows that M_0 cannot be treated by the methods of [9]. It states that the group $\text{Aut}(M_0)$ is amenable but the structure M_0 does not have a linear ordering so that the corresponding age has the order property and the Ramsey property.

It is worth noting that this statement already holds for the $\{E_1, E_2\}$ -reduct of M_0 , see the proof below. Thus our theorem also gives some interesting finitely homogeneous examples. On the other hand amenability of $\text{Aut}(M_0)$ is a harder task than the corresponding statement in the reduct's case.

The statement that $\text{Aut}(M_0)$ is amenable is a consequence of a stronger property, namely *Hrushovski's extension property* for partial isomorphisms. This is defined for Fraïssé limits as follows.

Definition 1.1 *A universal ultrahomogeneous structure U satisfies Hrushovski's extension property if for any finite family of finite partial isomorphisms between substructures of U there is a finite substructure $F < U$ containing these substructures so that any isomorphism from the family extends to an automorphism of F .*

Proposition 6.4 of [11] states that the structure U has Hrushovski's extension property if and only if $\text{Aut}(U)$ has a dense subgroup which is the union of a countable chain of compact subgroups. The latter implies amenability by Theorem 449C of [4].

Theorem 1.2 (a) *The structure M_0 satisfies Hrushovski's extension property. In particular the group $\text{Aut}(M_0)$ is amenable.*

(b) The structure M_0 does not have any expansion by a linear order so that $\text{Th}(M_0, <)$ admits elimination of quantifiers and the age of $(M_0, <)$ satisfies the Ramsey property.

The proof uses some material from [5]. We now describe it.

Let L be a finite relational language. We say that an L -structure F is *irreflexive* if for any $R \in L$, any tuple from F satisfying R consists of pairwise distinct elements. An irreflexive L -structure F is called a *link structure* if F is a singleton or F can be enumerated $\{a_1, \dots, a_n\}$ so that (a_1, \dots, a_n) satisfies a relation from L .

Let \mathcal{S} be a finite set of link structures. Then an L -structure N is of *link type* \mathcal{S} if any substructure of N which is a link structure is isomorphic to a structure from \mathcal{S} .

An L -structure F is *packed* if any pair from F belongs to a link structure which is a substructure of F .

If \mathcal{R} is a finite family of packed irreflexive L -structures, then an L -structure F is called *\mathcal{R} -free* if there does not exist a weak homomorphism (a map preserving the predicates) from a structure from \mathcal{R} to F .

Proposition 4 and Theorem 5 of [5] state that for any family of irreflexive link structures \mathcal{S} and any finite family of irreflexive packed L -structures \mathcal{R} the class of all irreflexive finite L -structures of link type \mathcal{S} which are \mathcal{R} -free, has the free amalgamation property and Hrushovski's extension property for partial isomorphisms.

We will use a slightly stronger version of this statement concerning *permorphisms*. A partial mapping ρ on U is called a **χ -permorphism**, if χ is a permutation of symbols in L preserving the arity and for every $R \in L$ and $\bar{a} \in \text{Dom}(\rho)$ we have

$$\bar{a} \in R \Leftrightarrow \rho(\bar{a}) \in R^\chi.$$

The following statement is a version of Lemma 6 from [5].

Lemma 1.3 *Let L be a finite language, χ_1, \dots, χ_n be arity preserving permutations of L and \mathcal{S} be a finite $\{\chi_i\}_{i \leq n}$ -invariant family of irreflexive link structures. Let \mathcal{R} be a finite family of finite irreflexive packed L -structures of link type \mathcal{S} so that \mathcal{R} is invariant under all χ_i . Let A be a finite structure which belongs to the class, say K , of L -structures of link type \mathcal{S} which are \mathcal{R} -free. Let ρ_i , $i \leq n$, be partial χ_i -permorphisms of A .*

Then there is a finite $B \in K$ containing A so that each ρ_i extends to a permutation of B which is a χ_i -permorphism.

Proof of Theorem 1.2. (a) For each $n > 0$ enumerate all E_n -classes. Consider the expansion of M_0 by distinguishing each E_n -class by a predicate $P_{n,i}$ according the enumeration. Let L^* be the language of all predicates $P_{n,i}$ and let M^* be the L^* -structure defined on M_0 . For every finite sublanguage $L' \subseteq L^*$ let $M^*(L')$ be the L' -reduct of M^* defined by these interpretations.

We denote by $\mathcal{K}(L')$ the class of all finite L' -structures with the properties that for any arity l represented by L' :

- any l -relation is irreflexive and invariant with respect to all permutations of variables,
- any two relations of L' of arity l have empty intersection.

Let $\mathcal{S}(L')$ be the set of all link structures of $\mathcal{K}(L')$ satisfying these two properties. Thus $\mathcal{K}(L')$ is of link type $\mathcal{S}(L')$.

Claim 1. For every finite sublanguage $L' \subseteq L^*$ the structure $M^*(L')$ is a universal structure with respect to the class $\mathcal{K}(L')$.

It is easy to see that any structure from $\mathcal{K}(L')$ can be expanded to a structure from \mathcal{K}_0 so that L' -predicates become classes of appropriate E_n 's.

Claim 2. For every finite sublanguage $L' \subseteq L^*$ the structure $M^*(L')$ is an ultra-homogeneous structure.

Let f be an isomorphism between finite substructures of $M^*(L')$. We may assume that $\text{Dom}(f)$ contains tuples representing all $M^*(L')$ -predicates of L' (some disjoint tuples can be added to $\text{Dom}(f)$ in a suitable way). Then f extends to an automorphism of M_0 fixing the classes of appropriate E_n 's which appear in L' . Thus this automorphism is an automorphism of $M^*(L')$ too.

Claim 3. For each finite sublanguage $L' \subseteq L^*$ let $\mathcal{R}(L')$ be the family of all packed L' -structures of the form $(\{a_1, \dots, a_n\}, P_{n,i}, P_{n,j})$, where $i \neq j$, $P_{n,i} = \{(a_1, \dots, a_n)\}$ and $P_{n,j} = \{(a_{\sigma(1)}, \dots, a_{\sigma(n)})\}$ for some permutation σ . Then the class $\mathcal{K}(L')$ is the class of all irreflexive finite L' -structures of link type $\mathcal{S}(L')$, which are $\mathcal{R}(L')$ -free.

The claim is obvious. By Proposition 4 and Theorem 5 of [5] we now see that $\mathcal{K}(L')$ is closed under substructures, has the joint embedding property, the free amalgamation property, Hrushovski's extension property and its version for permorphisms, i.e. the statement of Lemma 1.3.

By Claim 1 and Claim 2 the structure $M^*(L')$ is the universal homogeneous structure of $\mathcal{K}(L')$. In particular any tuple of finite partial isomorphisms (permorphisms) of $M^*(L')$ can be extended to a tuple of automorphisms (permorphisms) of a finite substructure of $M^*(L')$.

Note that the same statement holds for the structure M^* . To see this take any tuple f_1, \dots, f_k of finite partial isomorphisms (resp. χ_i -permorphisms) of M^* (assuming that χ_i are finitary). Let r be the size of the union $\bigcup_{i \leq k} \text{Dom}(f_i)$ and L' be the minimal (resp. $\{\chi_i\}_{i \leq k}$ -invariant) sublanguage of L^* of arity r containing of all relations of M^* which meet any tuple from $\bigcup_{i \leq k} \text{Dom}(f_i)$. Then there is a finite substructure A of $M^*(L')$ containing $\bigcup_{i \leq k} \text{Dom}(f_i)$ so that each f_i extends to an automorphism (resp. χ_i -permorphism) of A .

Let r' be the size of A . Let L'' be a sublanguage of L^* so that $L' \subseteq L''$ and for each arity $l \leq r'$ the sublanguage $L'' \setminus L'$ contains exactly one l -relation, say P_{l,n_l} (fixed by $\{\chi_i\}_{i \leq k}$). Since M^* is the universal homogeneous structure of $\mathcal{K}(L'')$ the substructure A can be chosen so that any l -subset of A which does not satisfy any relation from L' , does satisfy P_{l,n_l} .

As a result any automorphism (permorphism) of A preserves the relations of $L''' \setminus L'$ for any $L''' \subset L^*$ containing L'' . Thus it extends to an automorphism (permorphism) of $M^*(L''')$. In particular it extends to an automorphism (permorphism) of M^* .

As in Proposition 6.4 of [11] we see that $Aut(M^*)$ has a dense subgroup which is the union of a countable chain of compact subgroups. In particular we arrive at the following statement.

Claim 4. $Aut(M^*)$ is amenable.

Since each automorphism of M_0 is a permorphism of M^* and vice versa, we also see that $Aut(M_0)$ has a dense subgroup which is the union of a countable chain of compact subgroups. In particular $Aut(M_0)$ is amenable.

(b) Consider a linearly ordered expansion $(M_0, <)$ together with the corresponding age, say $\mathcal{K}^<$. Assume that $\mathcal{K}^<$ has the Ramsey property.

Note that $\mathcal{K}^<$ does not contain any three-element structure of the form $a < b < c$, where a and c belong to the same E_1 -class which is distinct from the E_1 -class of b . Indeed, otherwise repeating the argument of Theorem 6.4 from [9], we see that in any larger structure from $\mathcal{K}^<$ we can colour two-elements structures $a < b$ with $\neg E_1(a, b)$, so that there is no monochromatic three-element structure of the form above.

As a result we see that any E_1 -class of $(M_0, <)$ is convex. We now claim that the following structure B can be embedded into $(M_0, <)$.

Let $B = \{a_1 < a_2 < a_3 < a_4 < b_1 < b_2\}$, where the E_1 -classes of all elements are pairwise distinct, but the pairs $\{a_1, a_2\}$ and $\{b_1, b_2\}$ are E_2 -equivalent. We assume that in all other cases any two distinct pairs from B belong to distinct E_2 -classes. Moreover we assume that for each $k = 3, 4, 5$ all k -subsets from B belong to the same E_k -class. In particular the ordered structures defined on $\{a_1, a_2, a_3, a_4\}$ and $\{a_3, a_4, b_1, b_2\}$ are isomorphic. Let A represent this isomorphism class.

Since M_0 is the universal homogeneous structure with respect to \mathcal{K}_0 , taking any tuple $a'_1 < a'_2 < a'_3 < a'_4 < b'_1 < b'_2$ with pairwise distinct E_1 -classes we can find B in M_0 as a half of a copy of a structure from \mathcal{K}_0 consisting of 12 elements where each E_1 -class is represented by a pair (a'_i, a_i) or (b'_i, b_i) .

To show that the Ramsey property does not hold for the age of $(M_0, <)$ take any finite substructure C of this age which extends B . Fix any enumeration of E_2 -classes occurring in C . Then colour a copy of A red if the class of the first two elements is enumerated before the class of the last pair. Otherwise colour such a copy green. It is clear that C does not contain a structure isomorphic to B so that all substructures of type A are of the same colour. \square

Remark 1.4 It is worth noting that the class $\mathcal{K}_0^<$ of all linearly ordered members of \mathcal{K}_0 has JEP and AP, i.e. there is a generic expansion of M_0 by a linear ordering. To see AP we just apply the amalgamation described above together with the standard amalgamation of orderings.

2 Adding dense linear orders

In order to obtain a structure with the properties as in Section 1, but without Hrushovski's extension property we use a general approach from [3]. In fact our starting point is Corollary 2.8 from [3] that sets $\binom{M_0}{n}/E_n$ (definable in $Th^{eq}(M_0)$) are

stably embedded in M_0 .

We remind the reader that a 0-definable predicate P of a theory T is called **stably embedded** if every definable relation on P is definable with parameters from P . If M is a saturated model of T then P is stably embedded if and only if every elementary permutation of $P(M)$ extends to an automorphism of M (see remarks after Definition 2.4 in [3]). We now formulate Lemma 3.1 from [3].

Let T be a complete theory with two sorts S_0 and S_1 . Let \tilde{T}_1 be a complete expansion of $T \upharpoonright S_1$. Assume that S_1 is stably embedded. Then

- (1) $\tilde{T} = T \cup \tilde{T}_1$ is a complete theory;
- (2) S_1 is stably embedded in \tilde{T} and $\tilde{T} \upharpoonright S_1 = \tilde{T}_1$.
- (3) if T and \tilde{T}_1 are ω -categorical, then \tilde{T} is also ω -categorical.

We now describe our **variations** of M_0 . Let us fix $S_n = \binom{M_0}{n}/E_n$, $n \in \omega$, and consider them as a sequence of stably embedded sorts in $Th^{eq}(M_0)$ (this is Corollary 2.8 of [3]). We can distinguish relations $\{a_1, \dots, a_n\} \in e$, where $e \in S_n$ is an E_n -class, $n \in \omega$.

We also fix a subset $P \subset \omega \setminus \{1, 2\}$ and consider the language

$$L_P^S = \{E_n : 0 < n \in \omega\} \cup \{S_n, <_{S_n} : n \in P\},$$

where $<_{S_n}$ are binary relations on S_n . Let \tilde{T}_1 be the theory of sorts $\{S_n : n \in \omega\}$, where for every $n \in P$ the relation $<_{S_n}$ is a dense linear order without ends. When $n \notin P$ the sort S_n is considered as a pure set. This is an ω -categorical theory for each S_n . Applying Lemma 3.1 from [3] we define the complete theory $T_P^S = T_E \cup \tilde{T}_1$ which is ω -categorical and every sort S_n is stably embedded into T_P^S .

We now define an one-sorted version of T_P^S . Its countable model will be the example announced in Introduction.

Let $L_P = \{E_n : 0 < n \in \omega\} \cup \{<_n : n \in P\}$ be a first-order language, where each E_n and $<_n$ is a relational symbol of arity $2n$. The L_P -structure M is built by the Fraïssé's construction. Let us specify a class \mathcal{K}_P of finite L_P -structures, which will become the class of all finite substructures of M .

Assume that in each $C \in \mathcal{K}_P$ each relation $E_n(\bar{x}, \bar{y})$ determines an equivalence relation on the set (denoted by $\binom{C}{n}$) of unordered n -element subsets of C . As before for $C \in \mathcal{K}_P$ and $n > |C|$ we put that no $2n$ -tuple from C satisfies $E_n(\bar{x}, \bar{y})$.

For $n \in P$ the relations $<_n$ are irreflexive and respect E_n ,

$$\forall \bar{x}, \bar{y}, \bar{u}, \bar{w} (E_n(\bar{x}, \bar{y}) \wedge E_n(\bar{u}, \bar{w}) \wedge <_n(\bar{x}, \bar{u}) \rightarrow <_n(\bar{y}, \bar{w})).$$

Every $<_n$ is interpreted by a linear order on the set of E_n -classes. Therefore we take the corresponding axioms (assuming below that tuples consist of pairwise distinct elements):

$$\begin{aligned} & \forall \bar{x}, \bar{y} (<_n(\bar{x}, \bar{y}) \rightarrow \neg E_n(\bar{x}, \bar{y})); \\ & \forall \bar{x}, \bar{y}, \bar{z} (<_n(\bar{x}, \bar{y}) \wedge <_n(\bar{y}, \bar{z}) \rightarrow <_n(\bar{x}, \bar{z})); \\ & \forall \bar{x}, \bar{y} (\neg E_n(\bar{x}, \bar{y}) \rightarrow <_n(\bar{x}, \bar{y}) \vee <_n(\bar{y}, \bar{x})). \end{aligned}$$

Lemma 2.1 (1) *The class \mathcal{K}_P satisfies the joint embedding property and the amalgamation property.*

(2) *Let M be the generic structure of \mathcal{K}_P . For every $n > 0$ let $M_n = \binom{M}{n}/E_n$. Then $Th(M)$ is ω -categorical, admits elimination of quantifiers, and $<_n$ is a dense linear ordering on M_n without ends (when $n \in P$). The structure M is an expansion of M_0 .*

(3) *Let ρ_i , $i \leq k$, be a sequence of finitary maps on M_i which respect $<_i$ for $i \in P$. Then there is an automorphism $\alpha \in Aut(M)$ realising each ρ_i on its domain.*

Proof. (1) Given $A, B_1, B_2 \in \mathcal{C}$ with $B_1 \cap B_2 = A$, define $C \in \mathcal{K}$ as $B_1 \cup B_2$. The relations $E_n, <_n$, $n \leq |B_1 \cup B_2|$, are defined so that $C \in \mathcal{K}$, $B_1 < C$, $B_2 < C$ and the following conditions hold. Let $n \leq |B_1 \cup B_2|$. We put that all n -element n -tuples meeting both $B_1 \setminus B_2$ and $B_2 \setminus B_1$ are pairwise equivalent with respect to E_n . We additionally demand that they are equivalent to some tuple from some B_i , $i \in \{1, 2\}$, if $n \leq \max(|B_1|, |B_2|)$. If for some $i \in \{1, 2\}$, $|\binom{B_i}{n}/E_n| = 1$, then we put that all n -tuples $\bar{c} \in B_1 \cup B_2$ meeting B_i are pairwise E_n -equivalent. We additionally arrange that they are equivalent to some tuple from B_{3-i} if $n \leq |B_{3-i}|$. If $n \geq \max(|B_1|, |B_2|)$ then all n -element n -tuples from C are pairwise E_n -equivalent. We take E_n to be the minimal equivalence relation satisfying the conditions above. In particular if n -tuples \bar{b}_1 and \bar{b}_2 are E_n -equivalent to the same n -tuple from A , then $E_n(\bar{b}_1, \bar{b}_2)$.

We can now define the linear orderings $<_n$ on C/E_n for $n \in P$. There is nothing to do if $|\binom{C}{n}/E_n| = 1$. In the case when for some $i = 1, 2$, $|\binom{B_i}{n}/E_n| = 1$, the relation $<_n$ is defined by its restriction to B_{3-i} . When $|\binom{B_1}{n}/E_n| \neq 1 \neq |\binom{B_2}{n}/E_n|$ and V_1, V_2 is a pair of two $<_n$ -neighbours among E_n -classes having representatives both in $\binom{B_1}{n}$ and $\binom{B_2}{n}$, we amalgamate the $<_n$ -linear orderings between V_1 and V_2 assuming that all elements of $\binom{B_1}{n}/E_n \cap [V_1, V_2]$ are less than those from $\binom{B_2}{n}/E_n \cap [V_1, V_2]$.

We appropriately modify this procedure for intervals open from one side. It is clear that this defines $<_n$ -ordering on $\binom{C}{n}/E_n$.

(2) The statement that $Th(M)$ admits elimination of quantifiers and is ω -categorical, follows from (1). This also implies that M is a natural expansion of M_0 .

To see the second statement of this part of the lemma it is enough to show that for $n \in P$ and any two sequences $V_1 <_n \dots <_n V_k$ and $V'_1 <_n \dots <_n V'_k$ from M_n there is an automorphism of M taking each V_i to V'_i . To see this we use the fact that M is the Fraïssé limit of \mathcal{K}_P . This allows us to find pairwise disjoint representatives of classes V_1, \dots, V_k , say $\bar{a}_1, \dots, \bar{a}_k$, and classes V'_1, \dots, V'_k , say $\bar{a}'_1, \dots, \bar{a}'_k$, so that for every $m \neq n$ all m -tuples of the substructures $\bar{a}_1 \cup \dots \cup \bar{a}_k$ and $\bar{a}'_1 \cup \dots \cup \bar{a}'_k$ are E_m -equivalent. Moreover all n -tuples meeting at least two \bar{a}_s, \bar{a}_t or \bar{a}'_s, \bar{a}'_t also belong to a single E_n -class. Taking an appropriate isomorphism induced by these representatives we extend it to a required automorphism.

(3) We develop the argument of (2). For each ρ_i find a sequence $\bar{a}_1, \dots, \bar{a}_t$ of pairwise disjoint tuples from M representing the E_i -classes of the domain and of the range of ρ_i . We may assume that for any $j \neq i$ all j -tuples of the union $\Omega_i = \bar{a}_1 \cup \dots \cup \bar{a}_t$ belong to the same E_j -class. Moreover all i -tuples meeting at least two \bar{a}_l, \bar{a}_m also form a single E_i -class. Thus ρ_i can be realised by a partial map on Ω_i . We may

arrange that all Ω_i are pairwise disjoint and do not have common E_n -classes. Thus all ρ_i can be realised by a partial isomorphism on the union of these Ω_i . Since M is ultrahomogeneous, this partial isomorphism can be extended to an automorphism of M . \square

Let us consider M in the language L_P^S , i.e.

$$(M, E_1, \dots, E_n, \dots) \cup (M_1, *_1) \cup \dots \cup (M_n, *_n) \cup \dots,$$

where $*_n = <_n$ for $n \in P$ and disappears for $n \notin P$. By Lemma 2.1(3) the structure of all sorts $\{M_n : n \in \omega\}$ coincides with the theory \tilde{T}_1 of sorts $\{S_n : n \in \omega\}$ of the theory T_P^S . This implies the following corollary.

Corollary 2.2 *The theory of M in the language L_P^S coincides with T_P^S . In particular the sets M_n are stably embedded into M .*

We see that for $n \in P$ any automorphism of $(M_n, <_n)$ can be realized by an automorphism of M . Assume that $2n \notin P$. Let us consider automorphisms α of M_n which are *increasing*, i.e. for any $V \in M_n$, $V <_n \alpha(V)$.

Take an orbit of α of the following form:

$$\dots \rightarrow \bar{a}_{-1} \rightarrow \bar{a}_0 \rightarrow \bar{a}_1 \rightarrow \bar{a}_2 \rightarrow \bar{a}_3 \rightarrow \bar{a}_4 \rightarrow \dots$$

and consider E_{2n} -classes of tuples $\bar{a}_i \bar{a}_{i+1}$. Applying ultrahomogeneity and the choice of n it is easy to see that α can be taken so that there are four E_{2n} -classes, say V_1, V_2, V_3, V_4 , represented by consecutive pairs of tuples $\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4, \bar{a}_5, \bar{a}_6$ and α acts on them by $\mathbb{Z}/4\mathbb{Z}$:

$$\text{if } \bar{a}_1 \bar{a}_2 \in V_1, \text{ then } \bar{a}_2 \bar{a}_3 \in V_2, \bar{a}_3 \bar{a}_4 \in V_3 \text{ and } \bar{a}_4 \bar{a}_5 \in V_4,$$

where $\bar{a}_1 \bar{a}_2$ and $\bar{a}_5 \bar{a}_6$ are E_{2n} -equivalent.

Slightly generalising this situation we will say that a sequence $\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4, \bar{a}_5, \bar{a}_6$ is **$<_n$ -increasing of type $\mathbb{Z}/4\mathbb{Z}$** if the following conditions are satisfied:

- tuples $\bar{a}_1 \bar{a}_2$, $\bar{a}_2 \bar{a}_3$ and $\bar{a}_3 \bar{a}_4$ are of the same isomorphism type,
- tuples $\bar{a}_1 \bar{a}_2 \bar{a}_3 \bar{a}_4$ and $\bar{a}_3 \bar{a}_4 \bar{a}_5 \bar{a}_6$ are of the same isomorphism type and
- tuples $\bar{a}_1 \bar{a}_2$ and $\bar{a}_5 \bar{a}_6$ are E_{2n} -equivalent but not E_{2n} -equivalent to $\bar{a}_3 \bar{a}_4$.

Let L' be an extension of L_P and $M' = (M, \bar{\mathbf{r}})$ be an L' -expansion of M with quantifier elimination. We do not demand that $\bar{\mathbf{r}}$ is finite, we only assume that M' is a precompact expansion. It is clear that M' induces a subgroup of $\text{Aut}(M_n, <_n)$.

We will say that a sequence $\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4, \bar{a}_5, \bar{a}_6$ is **$<_n$ -increasing of type $\mathbb{Z}/4\mathbb{Z}$ in M'** if the definition above holds under the assumption that the isomorphism types appeared in the definition are considered with respect to the relations of M' .

Theorem 2.3 *Let M be the generic structure of \mathcal{K}_P where $P \neq \emptyset$. Then the group $G = \text{Aut}(M)$ is amenable, M does not satisfy Hrushovski's extension property and does not have an extremely amenable ultrahomogeneous expansion by a linear ordering.*

Let M' be a precompact expansion of M with quantifier elimination. If $\text{Aut}(M')$ is extremely amenable, then for any $n \in P$ with $2n \notin P$ the structure M' does not have an $<_n$ -increasing sequence of type $\mathbb{Z}/4\mathbb{Z}$.

The main point of this theorem is that although in different arities the structures induced by M are completely independent, any expansion M' as in the formulation simultaneously destroys M in all arities $n \in P$ with $2n \notin P$.

The proof below uses the proof of Theorem 1.2.

Proof of Theorem 2.3. For each $n > 1$ enumerate all E_n -classes. Consider the expansion of M by distinguishing each E_n -class by a predicate $P_{n,i}$ according the enumeration. Let L^* be the language of all predicates $P_{n,i}$ and let M^* be the L^* -structure defined on M . By Claims 1 - 4 of the proof of Theorem 1.2 the structure M^* has Hrushovski's extension property and $\text{Aut}(M^*)$ is amenable.

Let us consider the structure $(M_n, <_n)$, where $n \in P$. As it is isomorphic to $(\mathbb{Q}, <)$, the group $\text{Aut}(M_n, <_n)$ is extremely amenable ([9]).

Since each automorphism of M preserves all $<_i$, $i \in P$, it is easy to see that there is a natural homomorphism from $\text{Aut}(M)$ to the product

$$\prod_{i \in P} \text{Aut}(M_i, <_i) \times \prod_{i \notin P} \text{Sym}(M_i)$$

and $\text{Aut}(M^*)$ is the kernel of it. By Corollary 2.2 this homomorphism is surjective. Now by Theorem 449C of [4] we have the following claim.

The group $\text{Aut}(M)$ is amenable.

To see that M does not satisfy Hrushovski's extension property take $n \in P$ and let us consider any triple of pairwise disjoint n -tuples \bar{a} , \bar{b} , \bar{c} representing pairwise distinct elements of M_n so that

$$\bar{a} <_n \bar{b} <_n \bar{c}.$$

Then the map ϕ fixing \bar{a} and taking \bar{b} to \bar{c} cannot be extended to an automorphism of a finite substructure of M .

Consider a linearly ordered expansion $(M, <)$ with quantifier elimination. To see that $\text{Aut}(M, <)$ is not extremely amenable just apply the argument of statement (b) of Theorem 1.2. Since at arity 2 the structure M coincides with M_0 it works without any change.

To prove the second part of the theorem we slightly modify that argument.

Let $n \in P$ and $2n \notin P$. Let a structure B consist of $6n$ elements forming a sequence

$$\bar{a}_1 <_n \bar{a}_2 <_n \bar{a}_3 <_n \bar{a}_4 <_n \bar{b}_1 <_n \bar{b}_2,$$

where the tuples $\bar{a}_1\bar{a}_2$ and $\bar{b}_1\bar{b}_2$ are E_{2n} -equivalent but not of the same E_{2n} -class with $\bar{a}_3\bar{a}_4$. We assume that the tuples $\bar{a}_1\bar{a}_2$, $\bar{a}_2\bar{a}_3$, and $\bar{a}_3\bar{a}_4$ are of the same isomorphism class in M' and the substructure $\bar{a}_1\bar{a}_2\bar{a}_3\bar{a}_4 < M'$ is isomorphic to $\bar{a}_3\bar{a}_4\bar{b}_1\bar{b}_2 < M'$. Since

$\text{Aut}(M')$ is extremely amenable, these structures are rigid and the corresponding isomorphisms are uniquely defined on these tuples.

Let A represent the isomorphism class of $\bar{a}_1\bar{a}_2\bar{a}_3\bar{a}_4$ in M' . Let us show that the Ramsey property does not hold for the age of M' . Take any finite substructure C of this age which extends B . Fix any enumeration of E_{2n} -classes occurring in C . Then colour a copy of A red if the class of the first two n -tuples is enumerated before the class of the last pair. Otherwise colour such a copy green. It is clear that C does not contain a structure isomorphic to B so that all substructures of type A are of the same colour. \square

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